

# Lecture 13

## Multi-Junction Transmission Lines, Duality Principle

A simple extension of the transmission line theory of the previous lecture is to the case when transmission lines of different characteristic impedances and wave velocities are concatenated together. Myriads of devices can be designed using such admixture of transmission lines, transistors, and diodes.

Also, due to the symmetry of Maxwell's equations between the electric field and the magnetic field, once a set of solutions have been found for Maxwell's equations, new solutions can be found by symmetry arguments. This is known as duality principle in electromagnetics.

### 13.1 Multi-Junction Transmission Lines

The real world is usually more complex than the world of our textbooks. However, we need to distill problems in the real world into simpler problems that we can explain with our textbook examples. Figure 13.1 shows many real world technologies, but they can be approximated with transmission line models as shall be seen.

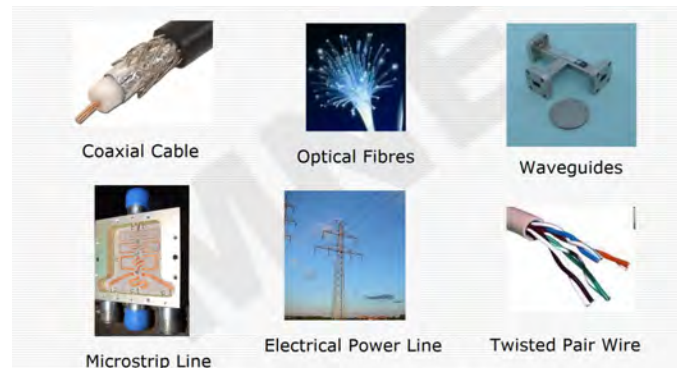


Figure 13.1: Different kinds of waveguides operating in different frequencies in power lines, RF circuits, microwave circuits, and optical fibers. Their salient physics or features can be captured or approximated by transmission lines (courtesy of Owen Casha).

An area where multi-junction transmission lines play an important role is in the microwave integrated circuit (MIC) area and the monolithic microwave integrated circuit (MMIC) area. An MMIC circuit is shown in Figure 13.2. Many of the components can be approximated by multi-junction transmission lines. They are clear motivation for studying this topic.

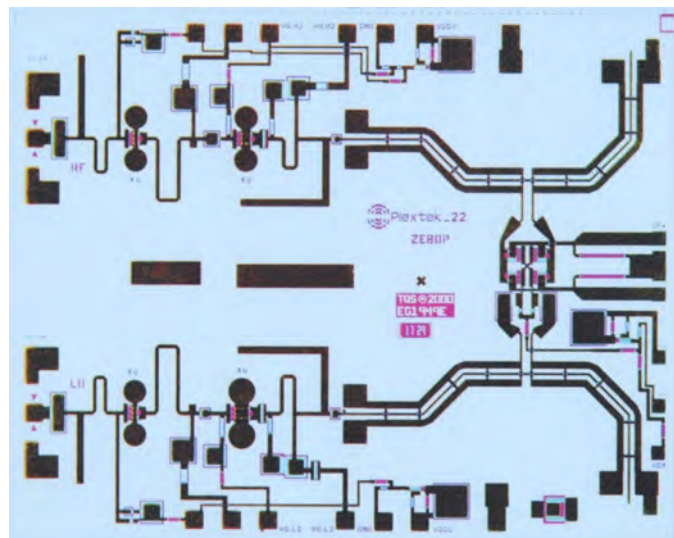


Figure 13.2: A generic GaAs monolithic microwave integrated circuit (MMIC). They are the motivation for studying multi-junction transmission lines (courtesy of Wikipedia).

By concatenating sections of transmission lines of different characteristic impedances, a large variety of devices such as resonators, filters, radiators, and matching networks can be formed. We will start with a single junction transmission line first. A good reference for such problem is the book by Collin [92], but much of the treatment here is not found in any textbooks.

### 13.1.1 Single-Junction Transmission Lines

Consider two transmission lines connected at a single junction as shown in Figure 13.3. For simplicity, we assume that the transmission line to the right is infinitely long so that a right-traveling wave is not reflected; namely, there is no reflected wave. And we assume that the two transmission lines have different characteristic impedances,  $Z_{01}$  and  $Z_{02}$ .

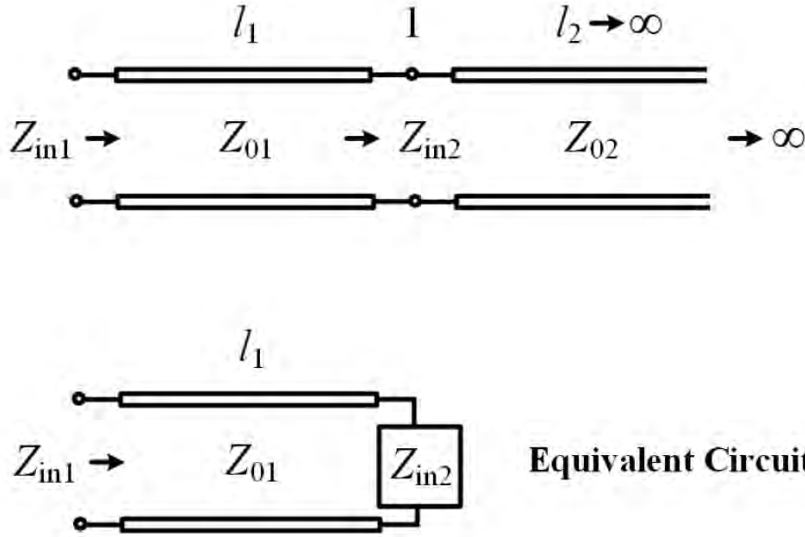


Figure 13.3: A single junction transmission line can be modeled by an equivalent transmission line terminated in a load  $Z_{in2}$ .

The impedance of the transmission line at junction 1 looking to the right, using the formula from previously derived,<sup>1</sup> is

$$Z_{in2} = Z_{02} \frac{1 + \Gamma_{L,\infty} e^{-2j\beta_2 l_2}}{1 - \Gamma_{L,\infty} e^{-2j\beta_2 l_2}} = Z_{02} \tag{13.1.1}$$

Since no reflected wave exists,  $\Gamma_{L,\infty} = 0$ , the above is just  $Z_{02}$ . As a result, transmission line 1 sees a load of  $Z_L = Z_{in2} = Z_{02}$  hooked to its end. The equivalent circuit is shown in Figure

<sup>1</sup>We should always remember that the relations between the reflection coefficient  $\Gamma$  and the normalized impedance  $Z_n$  are  $\Gamma = \frac{Z_n - 1}{Z_n + 1}$  and  $Z_n = \frac{1 + \Gamma}{1 - \Gamma}$ .

13.3 as well. Hence, we deduce that the reflection coefficient at junction 1 between line 1 and line 2, using the knowledge from the previous lecture, is  $\Gamma_{12}$ , and is given by

$$\Gamma_{12} = \frac{Z_L - Z_{01}}{Z_L + Z_{01}} = \frac{Z_{in2} - Z_{01}}{Z_{in2} + Z_{01}} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \quad (13.1.2)$$

where we have assumed that  $Z_L = Z_{in2} = Z_{02}$ .

### 13.1.2 Two-Junction Transmission Lines

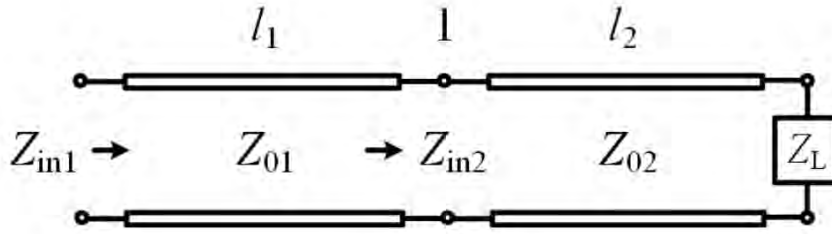


Figure 13.4: A single-junction transmission line with a load  $Z_L$  at the far (load) end of the second line. But it can be reduced to the equivalent circuit shown in the bottom of Figure 13.3.

Now, we look at the two-junction case. To this end, we first look at when line 2 is terminated by a load  $Z_L$  at its load end as shown in Figure 13.4. Then, using the formula derived in the previous lecture,

$$Z_{in2} = Z_{02} \frac{1 + \Gamma(-l_2)}{1 - \Gamma(-l_2)} = Z_{02} \frac{1 + \Gamma_{L2} e^{-2j\beta_2 l_2}}{1 - \Gamma_{L2} e^{-2j\beta_2 l_2}} \quad (13.1.3)$$

where we have used the fact that  $\Gamma(-l_2) = \Gamma_{L2} e^{-2j\beta_2 l_2}$ . It is to be noted that here, using knowledge from the previous lecture, that the reflection coefficient at the load end of line 2 is

$$\Gamma_{L2} = \frac{Z_L - Z_{02}}{Z_L + Z_{02}} \quad (13.1.4)$$

Now, line 1 sees a load of  $Z_{in2}$  hooked at its end. The equivalent circuit is the same as that shown in Figure 13.3. The generalized reflection coefficient at junction 1, which includes all the reflection of waves from its right, is now

$$\tilde{\Gamma}_{12} = \frac{Z_{in2} - Z_{01}}{Z_{in2} + Z_{01}} \quad (13.1.5)$$

Substituting (13.1.3) into (13.1.5), we have

$$\tilde{\Gamma}_{12} = \frac{Z_{02}(\frac{1+\Gamma}{1-\Gamma}) - Z_{01}}{Z_{02}(\frac{1+\Gamma}{1-\Gamma}) + Z_{01}} \quad (13.1.6)$$

where  $\Gamma = \Gamma_{L2}e^{-2j\beta_2 l_2}$ . The above can be rearranged to give

$$\tilde{\Gamma}_{12} = \frac{Z_{02}(1 + \Gamma) - Z_{01}(1 - \Gamma)}{Z_{02}(1 + \Gamma) + Z_{01}(1 - \Gamma)} \quad (13.1.7)$$

Finally, by further rearranging terms, after some somewhat tedious algebra, it can be shown that the above becomes

$$\tilde{\Gamma}_{12} = \frac{\Gamma_{12} + \Gamma}{1 + \Gamma_{12}\Gamma} = \frac{\Gamma_{12} + \Gamma_{L2}e^{-2j\beta_2 l_2}}{1 + \Gamma_{12}\Gamma_{L2}e^{-2j\beta_2 l_2}} \quad (13.1.8)$$

where  $\Gamma_{12}$ , the local reflection coefficient at the junction between line 1 and line 2, is given by (13.1.2), and  $\Gamma = \Gamma_{L2}e^{-2j\beta_2 l_2}$  is the general reflection coefficient<sup>2</sup> at  $z = -l_2$  due to the load  $Z_L$ . In other words,

$$\Gamma_{L2} = \frac{Z_L - Z_{02}}{Z_L + Z_{02}} \quad (13.1.9)$$

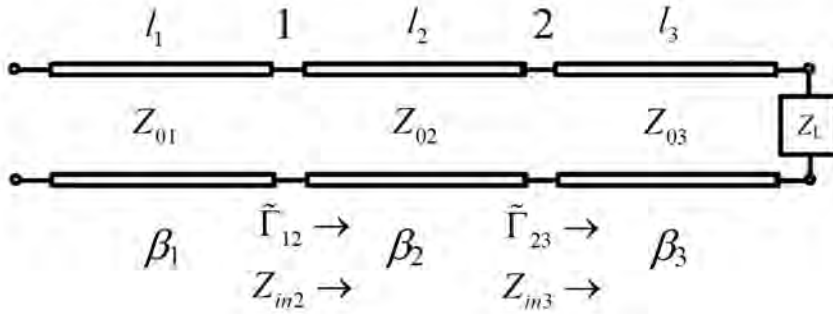


Figure 13.5: A two-junction transmission line with a load  $Z_L$  at the far end. The input impedance looking in from the far left can be found recursively using the formula (13.1.12) and (13.1.13).

Equation (13.1.8) is a **powerful formula** for multi-junction transmission lines. Imagine now that we add another section of transmission line as shown in Figure 13.5. We can use

<sup>2</sup>We will use the term “general reflection coefficient” at location  $z$  to mean the ratio between the amplitudes of the left-traveling wave and the right-traveling wave on a transmission line.

the aforementioned method to first find  $\tilde{\Gamma}_{23}$ , the generalized reflection coefficient at junction 2. Using formula (13.1.8), it is given by

$$\tilde{\Gamma}_{23} = \frac{\Gamma_{23} + \Gamma_{L3}e^{-2j\beta_3l_3}}{1 + \Gamma_{23}\Gamma_{L3}e^{-2j\beta_3l_3}} \quad (13.1.10)$$

where  $\Gamma_{L3}$  is the load reflection coefficient due to the load  $Z_L$  hooked to the end of transmission line 3 as shown in Figure 13.5. Here, it is given as

$$\Gamma_{L3} = \frac{Z_L - Z_{03}}{Z_L + Z_{03}} \quad (13.1.11)$$

Given the knowledge of  $\tilde{\Gamma}_{23}$ , we can use (13.1.8) again to find the new  $\tilde{\Gamma}_{12}$  at junction 1. It is now

$$\tilde{\Gamma}_{12} = \frac{\Gamma_{12} + \tilde{\Gamma}_{23}e^{-2j\beta_2l_2}}{1 + \Gamma_{12}\tilde{\Gamma}_{23}e^{-2j\beta_2l_2}} \quad (13.1.12)$$

The equivalent circuit is again that shown in Figure 13.3. Therefore, we can use (13.1.12) recursively to find the generalized reflection coefficient for a multi-junction transmission line. Once the reflection coefficient is known, the impedance at that location can also be found. For instance, at junction 1, the impedance is now given by

$$Z_{in2} = Z_{01} \frac{1 + \tilde{\Gamma}_{12}}{1 - \tilde{\Gamma}_{12}} \quad (13.1.13)$$

instead of (13.1.3). In the above,  $Z_{01}$  is used because the generalized reflection coefficient  $\tilde{\Gamma}_{12}$  is the total reflection coefficient for an incident wave from transmission line 1 that is sent toward the junction 1. Previously,  $Z_{02}$  was used in (13.1.3) because the reflection coefficients in that equation was for an incident wave sent from transmission line 2.

If the incident wave were to have come from line 2, then one can write  $Z_{in2}$  as

$$Z_{in2} = Z_{02} \frac{1 + \tilde{\Gamma}_{23}e^{-2j\beta_2l_2}}{1 - \tilde{\Gamma}_{23}e^{-2j\beta_2l_2}} \quad (13.1.14)$$

With some algebraic manipulation, it can be shown that (13.1.13) and (13.1.14) are identical. But (13.1.13) is closer to an experimental scenario where one measures the reflection coefficient by sending a wave from line 1 with no knowledge of what is to the right of junction 1.

Transmission lines can be made easily in microwave integrated circuit (MIC) by etching or milling. A picture of a microstrip line waveguide or transmission line is shown in Figure 13.6.

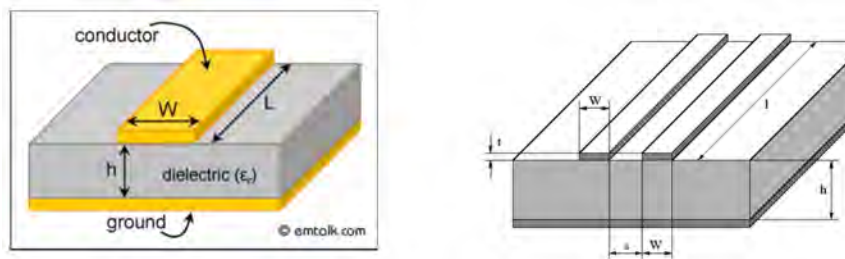


Figure 13.6: Schematic of a microstrip line with the signal line above, and a ground plane below (left). A strip line with each strip carrying currents of opposite polarity (right). A ground plane is not needed in the second case.

### 13.1.3 Stray Capacitance and Inductance

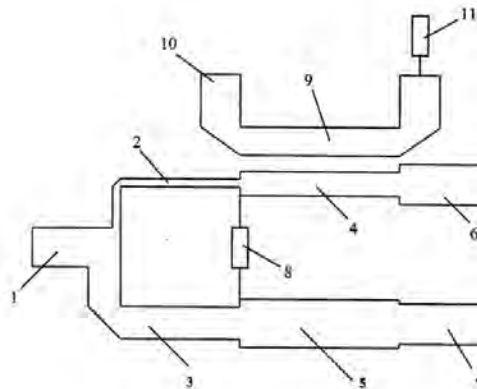


Figure 13.7: A general microwave integrated circuit with different kinds of elements.

The junction between two transmission lines is not as simple as we have assumed. In the real world, or in MIC, the waveguide junction has discontinuities in line width, or shape. This can give rise to excess charge cumulation or constricted current flow. Excess charge gives rise to excess electric field which corresponds to excess electric stored energy. This can be modeled by stray or parasitic capacitances.

Alternatively, there could be excess or constricted current flow that give rise to excess magnetic field. Excess magnetic field gives rise to excess magnetic stored energy. This can be modeled by stray or parasitic inductances. Hence, a junction can be approximated by a circuit model as shown in Figure 13.8 to account for these effects. The Smith chart or the method we have outlined above can still be used to solve for the input impedances of a

transmission circuit when these parasitic circuit elements are added.

Notice that when the frequency is zero or low, these stray capacitances and inductances are negligible, we retrieve the simple junction model. But since their impedance and admittance are  $j\omega L_s$  and  $j\omega C_s$ , respectively, they are non-negligible and are instrumental in modeling high frequency circuits.

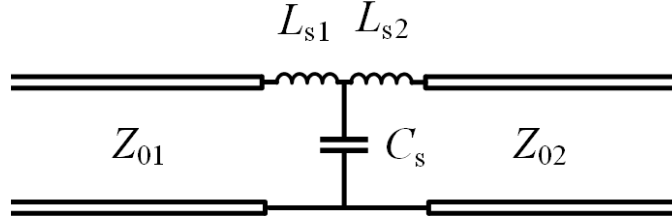


Figure 13.8: A junction between two microstrip lines can be modeled with stray junction capacitance and stray inductances. The capacitance is used to account for excess charges at the junction, while the inductances model the excess current at the junction. They are important as the frequency increases.

## 13.2 Duality Principle

Duality principle exploits the inherent symmetry of Maxwell's equations. Once a set of  $\mathbf{E}$  and  $\mathbf{H}$  fields has been found to solve Maxwell's equations for a certain geometry, another set for a similar geometry can be found by invoking this principle. Maxwell's equations in the frequency domain, including the fictitious magnetic sources, are

$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = -j\omega \mathbf{B}(\mathbf{r}, \omega) - \mathbf{M}(\mathbf{r}, \omega) \quad (13.2.1)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, \omega) = j\omega \mathbf{D}(\mathbf{r}, \omega) + \mathbf{J}(\mathbf{r}, \omega) \quad (13.2.2)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, \omega) = \varrho_m(\mathbf{r}, \omega) \quad (13.2.3)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, \omega) = \varrho(\mathbf{r}, \omega) \quad (13.2.4)$$

One way to make Maxwell's equations invariant is to do the following substitutions.

$$\mathbf{E} \rightarrow \mathbf{H}, \quad \mathbf{H} \rightarrow -\mathbf{E}, \quad \mathbf{D} \rightarrow \mathbf{B}, \quad \mathbf{B} \rightarrow -\mathbf{D} \quad (13.2.5)$$

$$\mathbf{M} \rightarrow -\mathbf{J}, \quad \mathbf{J} \rightarrow \mathbf{M}, \quad \varrho_m \rightarrow -\varrho, \quad \varrho \rightarrow \varrho_m \quad (13.2.6)$$

The above swaps retain the right-hand rule for plane waves. When material media is included, such that  $\mathbf{D} = \bar{\epsilon} \cdot \mathbf{E}$ ,  $\mathbf{B} = \bar{\mu} \cdot \mathbf{H}$ , for anisotropic media, Maxwell's equations become

$$\nabla \times \mathbf{E} = -j\omega \bar{\mu} \cdot \mathbf{H} - \mathbf{M} \quad (13.2.7)$$

$$\nabla \times \mathbf{H} = j\omega \bar{\epsilon} \cdot \mathbf{E} + \mathbf{J} \quad (13.2.8)$$



$$\nabla \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H} = \varrho_m \quad (13.2.9)$$

$$\nabla \cdot \bar{\boldsymbol{\varepsilon}} \cdot \mathbf{E} = \varrho \quad (13.2.10)$$

In addition to the above swaps, one need further to swap the material parameters, namely,

$$\bar{\boldsymbol{\mu}} \rightarrow \bar{\boldsymbol{\varepsilon}}, \quad \bar{\boldsymbol{\varepsilon}} \rightarrow \bar{\boldsymbol{\mu}} \quad (13.2.11)$$

### 13.2.1 Unusual Swaps<sup>3</sup>

There are other swaps where seemingly the right-hand rule is not preserved, e.g.,

$$\mathbf{E} \rightarrow \mathbf{H}, \mathbf{H} \rightarrow \mathbf{E}, \mathbf{M} \rightarrow -\mathbf{J}, \mathbf{J} \rightarrow -\mathbf{M}, \quad (13.2.12)$$

$$\varrho_m \rightarrow -\varrho, \varrho \rightarrow -\varrho_m, \bar{\boldsymbol{\mu}} \rightarrow -\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\varepsilon}} \rightarrow -\bar{\boldsymbol{\mu}} \quad (13.2.13)$$

The above swaps will leave Maxwell's equations invariant, but when applied to a plane wave, the right-hand rule seems violated.

The deeper reason is that solutions to Maxwell's equations are not unique, since there is a time-forward as well as a time-reverse solution. In the frequency domain, this shows up in the choice of the sign of the  $\mathbf{k}$  vector where in a plane wave  $k = \pm\omega\sqrt{\mu\varepsilon}$ . When one does a swap of  $\mu \rightarrow -\varepsilon$  and  $\varepsilon \rightarrow -\mu$ ,  $k$  is still indeterminate, and one can always choose a root where the right-hand rule is retained.

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<sup>3</sup>This section can be skipped on first reading.

### 13.3 Fictitious Magnetic Currents

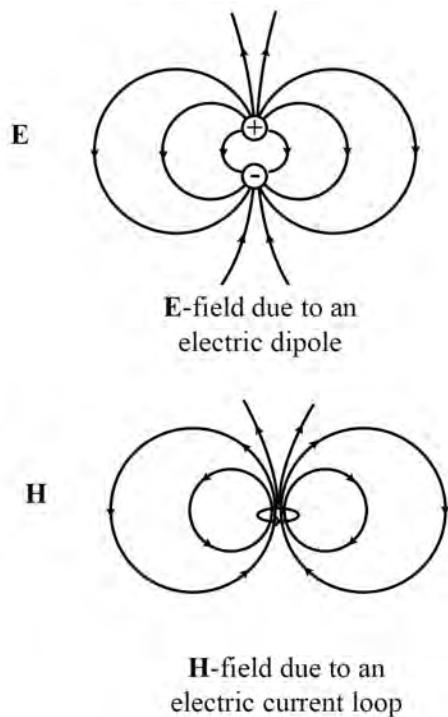


Figure 13.9: Sketches of the electric field due to an electric dipole and the magnetic field due to a electric current loop. The  $\mathbf{E}$  and  $\mathbf{H}$  fields have the same pattern, and can be described by the same formula.

Even though magnetic charges or monopoles do not exist, magnetic dipoles do. For instance, a magnet can be regarded as a magnetic dipole. Also, it is believed that electrons have spins, and these spins make electrons behave like tiny magnetic dipoles in the presence of a magnetic field.

Also if we form electric current into a loop, it produces a magnetic field that looks like the electric field of an electric dipole. This resembles a magnetic dipole field. Hence, a magnetic dipole can be made using a small electric current loop (see Figure 13.9). Because of these similarities, it is common to introduce fictitious magnetic charges and magnetic currents into Maxwell's equations. One can think that these magnetic charges always occur in pair and together. Thus, they do not contradict the absence of magnetic monopole.

The electric current loops can be connected in series to make a toroidal antenna as shown in Figure 13.10. The toroidal antenna is used to drive a current in an electric dipole. Notice that the toroidal antenna resembles the primary winding of a transformer circuit. In essence,

the toroidal loops, which mimic a magnetic current loop, produces an electric field that will drive current through the electric dipole. This is dual to the fact that an electric current loop produces a magnetic field.

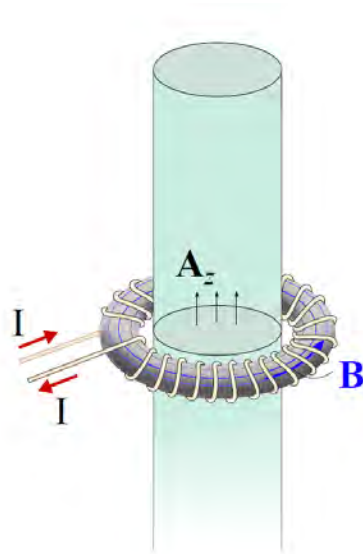


Figure 13.10: A toroidal antenna used to drive an electric current through a conducting cylinder of a dipole. It works similarly to a transformer: one can think of them as the primary and secondary turns (windings) of a transformer (courtesy of Q. S. Liu [93]).

